Mathemat[illegible]. vol 2

THE

ANTECEDENTAL CALCULUS,

OR A

GEOMETRICAL METHOD OF REASONING,

WITHOUT ANY

CONSIDERATION OF MOTION OR VELOCITY

APPLICABLE TO EVERY PURPOSE, TO WHICH FLUXIONS HAVE BEEN OR CAN BE APPLIED;

WITH THE

GEOMETRICAL PRINCIPLES OF INCREMENTS, &c.

AND THE

CONSTRUCTIONS OF SOME PROBLEMS

AS A

FEW EXAMPLES SELECTED FROM AN ENDLESS AND INDEFINITE VARIETY OF THEM RESPECTING SOLID GEOMETRY, WHICH HE HAS BY HIM IN MANUSCRIPT.

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LONDON:

PRINTED FOR G. G. J. AND J. ROBINSON, PATERNOSTER-ROW,

1793.

HAVING, in a Paper, read before the Royal Society, the 6th of March, 1777, and publiſhed in the Philoſophical Tranſactions of that year, promiſed to deliver, without any conſideration of Motion or Velocity, a Geometrical Method of Reaſoning applicable to every purpoſe, to which the much celebrated Doctrine of Fluxions of the illuſtrious NEWTON has been, or can be, applied; and having taken notice of the ſame Method, in a ſmall Performance, written in Latin, and printed the 16th of July, 1776, I now proceed to fulfil my promiſe with as much conciſeneſs as perſpicuity and preciſion will admit of.

If in Formula 1ſt, in Theorem 3, of my Univerſal Compariſon, (when R is to Q as 2 to 1), for A there be ſubſtituted $A+N$, we get this geometrical expreſſion $A+N+\overline{A+N}\cdot\frac{A-B+N}{B}$, which is equal to theſe two $A+A\cdot\frac{A-B}{B}$ and $\frac{2AN+N^2}{B}$. Whence it is manifeſt that the exceſs of the magnitude, which has to B the duplicate ratio of $A+N$ to B, above the magnitude, that has to B the duplicate ratio of A to B, is expreſſed by the magnitude of the ſame kind $\frac{2AN+N^2}{B}$. In like manner the exceſs of the magnitude, which has to B the triplicate ratio of $A+N$ to B, above the magnitude, that has to B the triplicate ratio of A to B, is geometrically expreſſed by $\frac{3A^2N+3AN^2+N^3}{B^2}$, (R being then to Q as 3 to 1). And, in general, the exceſs of the magnitude, which has to B a ratio having to the ratio of $A+N$ to B the ratio of R to Q (when R has to Q any given ratio whatever), above the magnitude, which has to B a ratio having to the ratio of A to B the ſame ratio of R to Q, is geometrically expreſſed by

$$\frac{\frac{R}{Q}\cdot A^{\frac{R-Q}{Q}}\cdot N+\frac{R}{Q}\cdot\frac{R-Q}{2Q}\cdot A^{\frac{R-2Q}{Q}}\cdot N^2+\frac{R}{Q}\cdot\frac{R-Q}{2Q}\cdot\frac{R-2Q}{3Q}\cdot A^{\frac{R-3Q}{Q}}\cdot N^3+\&c.}{B^{\frac{R-Q}{Q}}}.$$

Preciſely in the ſame manner does it appear, that the exceſs of the magnitude, which has to B a ratio, having to the ratio of A to B the ratio of R to Q, above the magnitude, which has to B a ratio, having to the ratio of A—N to B the ratio of R to Q, is geometrically expreſſed by

$$\frac{\frac{R}{Q}\cdot A^{\frac{R-Q}{Q}}\cdot N-\frac{R}{Q}\cdot\frac{R-Q}{2Q}\cdot A^{\frac{R-2Q}{Q}}\cdot N^2+\frac{R}{Q}\cdot\frac{R-Q}{2Q}\cdot\frac{R-2Q}{3Q}\cdot A^{\frac{R-3Q}{Q}}\cdot N^3-+-\&c.}{B^{\frac{R-Q}{Q}}}.$$

But if $A + N$ and $A - N$ ſtand to B in relations nearer to that of equality than by any given or aſſigned magnitude of the ſame kind, theſe general expreſſions become $\frac{\frac{R}{Q}.A^{\frac{R-Q}{Q}}.N}{B^{\frac{R-Q}{Q}}}$. This I call the antecedental of the magnitude which has to B ſuch a ratio as has to the ratio of $\dot{A}$ to B the ratio of R to Q. Now if N the antecedental of A be denoted by $\dot{A}$ or $\overset{\bullet}{A}$ (for the notation does not alter the nature of the caſe one iota) it becomes $\frac{\frac{R}{Q}.A^{\frac{R-Q}{Q}}.\dot{A}}{B^{\frac{R-Q}{Q}}}$ or $\frac{\frac{R}{Q}.A^{\frac{R-Q}{Q}}.\overset{\bullet}{A}}{B^{\frac{R-Q}{Q}}}$, having to $\dot{A}$ or $\overset{\bullet}{A}$ the ratio of $\frac{\frac{R}{Q}.A^{\frac{R-Q}{Q}}}{B^{\frac{R-Q}{Q}}} : 1$. If $Q = 1$, and $R = 2, 3, 4, 5$, &c. this expreſſion gives $\frac{2A\dot{A}}{B}$, $\frac{3A^2\dot{A}}{B^2}$, $\frac{4A^3\dot{A}}{B^3}$, $\frac{5A^4\dot{A}}{B^4}$, &c. reſpectively. And if $R = 1$, and $Q = 2, 3, 4, 5$, &c, it gives $\frac{A^{-\frac{1}{2}}.\dot{A}}{2B^{-\frac{1}{2}}}$, $\frac{A^{-\frac{2}{3}}.\dot{A}}{3B^{-\frac{2}{3}}}$, $\frac{A^{-\frac{3}{4}}.\dot{A}}{4B^{-\frac{3}{4}}}$, $\frac{A^{-\frac{4}{5}}.\dot{A}}{5B^{-\frac{4}{5}}}$, &c. reſpectively.

The ſame otherwiſe.

From Formula 5th, Theorem 3, it is evident that the exceſs of the magnitude (having to B ſuch a ratio as has to the ratio of A to B the ratio of R to Q) above B, if A ſtand to B in a relation nearer to that of equality than by any given or aſſigned magnitude of the ſame kind is expreſſed by $\frac{R}{Q}.\overline{A-B}$ or by $\frac{R}{Q}.\dot{A}$ or $\frac{R}{Q}.\overset{\bullet}{A}$ (if in this caſe A—B be denoted by $\dot{A}$ or $\overset{\bullet}{A}$); which expreſſion is always as the meaſure or quantity of the ratio, that the ſaid Formula has to B, (whatever be the relation of A to B). For in the ſimple, duplicate, triplicate, quadruplicate, &c. ratios, it gives $\dot{A}$, $2\dot{A}$, $3\dot{A}$, $4\dot{A}$, &c. reſpectively; and in the ſubduplicate, ſubtriplicate, ſubquadruplicate, &c. $\frac{\dot{A}}{2}$, $\frac{\dot{A}}{3}$, $\frac{\dot{A}}{4}$, &c. Now, if the ratio of $\frac{R}{Q}.\dot{A}$ to $\dot{A}$ denoting the relation between the quantities of the ratios, which the magnitude expreſſed by this Formula, and A have reſpectively to B, be compounded with the ratio of theſe magnitudes themſelves,

themselves, when A has to B any ratio whatever, that is, of $\frac{A^{\frac{R}{Q}}}{B^{\frac{R-Q}{Q}}}$ to A, we get the ratio of their antecedentals equal to $\frac{R}{Q}$. $\frac{A^{\frac{R}{Q}}}{B^{\frac{R-Q}{Q}}}$. $\dot{A}$ to A. $\dot{A}$ or to $\frac{R}{Q}$. $\frac{A^{\frac{R-Q}{Q}}.\dot{A}}{B^{\frac{R-Q}{Q}}}$ to $\dot{A}$, and the antecedental of the magnitude which has to B such a ratio as has to the ratio of A to B the ratio of R to Q, becomes the same geometrical expression as before. Whence it is manifest that the antecedentals of the magnitudes A, A. $\frac{A}{B}$, A. $\frac{A^2}{B^2}$, &c, . . A. $\frac{A^{\frac{R-Q}{Q}}}{B^{\frac{R-Q}{Q}}}$ are to each other in ratios compounded of the ratios of these magnitudes to each other respectively, and the ratios of magnitudes to each other, which are as the measures or quantities of the ratios, that A, A. $\frac{A}{B}$, A. $\frac{A^2}{B^2}$, &c. have respectively to the given magnitude B, or common geometrical standard of comparison. Consequently the ratio of the antecedental of any such magnitude as $\frac{A^{\frac{R}{Q}}}{B^{\frac{R-Q}{Q}}}$ to B is compounded of the ratio of this magnitude to A (the antecedent of the ratio of A to B) with the ratio, which a magnitude (that is as the measure or quantity of the ratio of $\frac{A^{\frac{R}{Q}}}{B^{\frac{R-Q}{Q}}}$ to B) has to B the common consequent of the ratios.

In Theorem 1, (which refers to the composition of ratios, whether the magnitudes in the different ratios be homogeneous, or heterogeneous, provided that taken two and two, from the first inclusively, they be magnitudes of the same, but of any kind), if for A, C, E, &c. there be substituted A ± M, C ± N, E ± P, &c, in Formula 1st, we get $\pm M +$

$\frac{A.\overline{\overline{C-D\pm N}-\overline{C-D}}\pm M.\overline{C-D\pm N}}{D}$ + &c. to $(\frac{p-1}{1})$ terms,

$+\frac{A.\overline{\overline{C-D\pm N}.\overline{E-F\pm P}-\overline{C-D}.\overline{E-F}}\pm M.\overline{C-D\pm N}.\overline{E-F\pm P}}{D.F}$ + &c, to $(\frac{p-1}{1}.\frac{p-2}{2})$ terms, + &c. &c. expressing geometrically both the excess of the magnitude, (which has to B the ratio that is produced by compounding the ratios of

C+N

$C+N$ to D, $E+P$ to F, &c. continued to the number $(\frac{p-1}{1})$ with the ratio of $A+M$ to B) above the magnitude denoted by this Formula, viz. when M, N, P, &c, are taken with the ſign + before them; and the exceſs of the magnitude expreſſed by this Formula above that, which has to B the ratio that is produced by compounding the ratios of $C-N$ to D, $E-P$ to F, &c. continued to the ſame number $(\frac{p-1}{1})$ with the ratio of $A-M$ to B, viz. when M, N, P, &c, &c. are taken with the ſign — before them. But in both caſes this expreſſion if $A\pm M$, $C\pm N$, $E\pm P$, &c. ſtand to A, C, E, &c. reſpectively, in relations nearer to that of equality than by any given or aſſigned magnitudes of the ſame kind, and only the two firſt ratios are compounded gives $\frac{A.N+C.M}{D}$ or $\frac{A.\dot{C}+C.\dot{A}}{D}$ (if M, N, P, &c, be denoted by $\dot{A}$, $\dot{C}$, $\dot{E}$, &c. or $\mathring{A}$, $\mathring{C}$, $\mathring{E}$, &c.) for the antecedental of $\frac{A.C}{D}$; and when the three firſt ratios are compounded, $\frac{A.C.\dot{E}+A.E.\dot{C}+C.E.\dot{A}}{D.F}$ for the antecedental of $\frac{A.C.E}{D.F}$, and ſo on.

In Theorem 2, which refers to the decompoſition of ratios, if for A, C, E, &c. there be ſubſtituted $A\pm M$, $C\pm N$, $E\pm P$, &c. we get in Formula 2d, (if $A\pm M$, $C\pm N$, &c. ſtand to A, C, &c. reſpectively, in relations nearer to that of equality than by any given or aſſigned magnitudes of the ſame kind, and the ſecond ratio is decompounded with the firſt) $\frac{C.D.M-A.D.N}{C^2}$ or $\frac{CD\dot{A}-AD\dot{C}}{C^2}$, (if for M, N, &c. there be ſubſtituted $\dot{A}$, $\dot{C}$, &c. or $\mathring{A}$, $\mathring{C}$, &c.) for the antecedental of $\frac{AD}{C}$, and ſo on for any number of ratios.

The Antecedents and their Antecedentals will therefore ſtand thus, B being the ſtandard of compariſon.

Antecedent.		Antecedental.
A	- -	$\dot{A}$ or $\mathring{A}$.
$A\frac{A}{B}$	- -	$\frac{2A\dot{A}}{B}$ or $\frac{2A\mathring{A}}{B}$ or $2\dot{A}+\frac{2M\dot{A}}{B}$ (putting M for A—B), or &c.
A^2 or $A.\frac{A}{B}.B.$	- -	$2A\dot{A}$ or $2B\dot{A}+2M\dot{A}$, or &c.

$A.\frac{A^2}{B^2}$

Antecedent.	Antecedental.
$A.\frac{A^2}{B^2}$ - -	$\frac{3A^2\dot{A}}{B^2}$ or $\frac{3A\dot{A}}{B} + \frac{3AM\dot{A}}{B^2}$ or $3\dot{A} + \frac{6M\dot{A}}{B} + \frac{3M^2\dot{A}}{B^2}$, or &c.
A^3 or $A.\frac{A^2}{B^2}.B^2$ -	$3A^2\dot{A}$, or $3BA\dot{A} + 3AM\dot{A}$, or $3B^2\dot{A} + 6BM\dot{A} + 3M^2\dot{A}$, or &c.

$$\text{And in general}\left.\right\}\ \frac{A^{\frac{R}{Q}}}{B^{\frac{R-Q}{Q}}}\left\{\begin{array}{l}\dfrac{RA^{\frac{R-Q}{Q}}.\dot{A}}{QB^{\frac{R-Q}{Q}}}\\[2ex] \text{or } \dfrac{RA\dot{A}}{QB} + R.\dfrac{R-2Q}{Q}.\dfrac{AM\dot{A}}{QB^2} + R.\dfrac{R-2Q}{Q}.\dfrac{R-3Q}{2Q}.\dfrac{AM^2\dot{A}}{QB^3}. + \text{\&c.}\\[2ex] \text{or } \dfrac{R\dot{A}}{Q} + R.\dfrac{R-Q}{Q}.\dfrac{M\dot{A}}{QB} + R.\dfrac{R-Q}{Q}.\dfrac{R-2Q}{2Q}.\dfrac{M^2\dot{A}}{QB^2} + \text{\&c.}\\[1ex] \text{or \&c. \&c.}\end{array}\right.$$

$A\frac{C}{D}$ - - $\frac{A\dot{C}+C\dot{A}}{D}$, or &c, &c.

AC - - $A\dot{C} + C\dot{A}$, or &c. &c.

$A.\frac{CE}{DF}$ - - $\frac{AC\dot{E}+AE\dot{C}+CE\dot{A}}{DF}$, or &c. &c.

ACE - - - $AC\dot{E} + AE\dot{C} + CE\dot{A}$, or &c. &c.

and ſo on.

$A.\frac{D}{C}$ - - - $\frac{CD\dot{A}-AD\dot{C}}{C^2}$, or &c. &c.

$\frac{A}{C}$ - - - $\frac{C\dot{A}-A\dot{C}}{C^2}$, or &c. &c. if the expreſſion be ſuppoſed to become numerical and the given magnitude D to be 1.

BA - - - $B\dot{A}$.

$\frac{DA^n}{B^n}$ - - - $\frac{nDA^{n-1}\dot{A}}{B^n}$, or &c.

$B+A+C$ - $\dot{A}+\dot{C}$.

$BA + \frac{DA^n}{B^n}$ - - $B\dot{A} + \frac{nDA^{n-1}\dot{A}}{B^n}$, or &c.

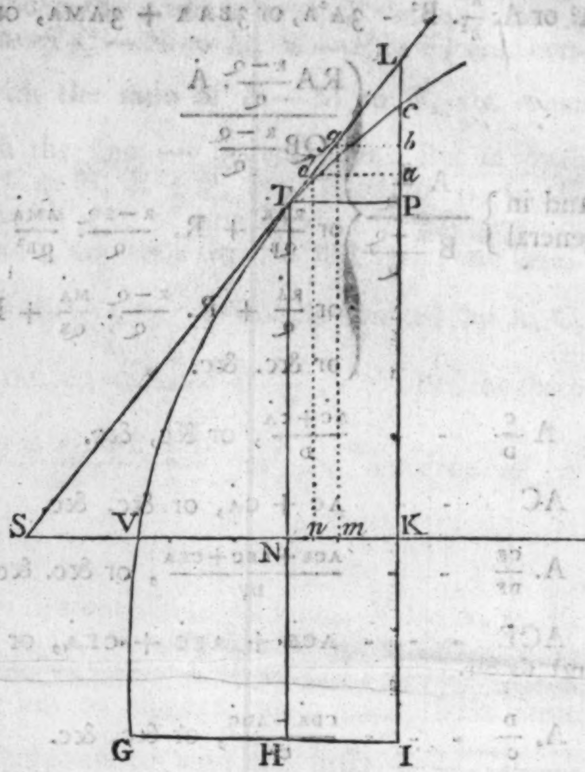
and ſo on.

When any magnitude is given, and undergoes no augmentation or diminution, increaſe or decreaſe in the compoſition or decompoſition of ratios, it has no antecedental. And in any caſe when the ratio is the greateſt or leaſt poſſible that the caſe admits of, the antecedent can have, in reality, no antecedental,

 and

and the expressions denoting them, in such cases, must be equivalent to nothing. In a similar manner are derived the second, third, &c. Antecedentals, as well as the Geometrical Principles of what is vulgarly called the Exponential Calculus.

Let V T c be any curve, V N its abscis, N T the corresponding semi-ordinate perpendicular thereto, S T L a tangent to the curve at the point T, L I and V G parallel to T N and G I, T P parallel to V K. Then it is manifest that, if V K be to V N in a relation nearer to that of equality than by any given or assigned magnitude of the same kind, or than any given or assigned ratio, K L to N T, and the right line equal to V T and T L together to the right line equal to V T, are also in relations nearer to that of equality than by any given or assigned magnitudes of the same kind, or than any given or assigned ratios.



For, if this be denied, let b P be the given or assigned magnitude by which K L exceeds T N, and K b to T N the given or assigned ratio it has to T N, whilst the ratio of V K to V N is nearer to that of equality than by any given or assigned magnitude of the same kind, or than any given or assigned ratio; and let $g\,b$, $d\,a$ be parallel to T P, and $g\,m$, $d\,n$ be parallel to T N.

Then, since V K is expressed by V m, if K L be expressed by K b or $m\,g$, since the triangles S K L, S N T, are always similar, and the ratios S K to S N, and K L to N T always equal, and the ratio of $g\,m$ to T N is given or assigned, the ratio of S m to S N is also given or assigned. Wherefore (6. Eu. Data) the ratio of S m to N m is given, and N m is a given or assignable magnitude contrary to the hypothesis.

The same may be demonstrated by a continual subdivision of L P, which will give a P less than the given or assigned magnitude b P, however small it be, and N n corresponding thereto, less than N m.

When the ratio of V K to V N is therefore nearer to that of equality than by any given or assigned magnitude of the same kind, or than any given or assigned ratio, the ratios of K L to N T, V T + T L to V T, V T c to V T, S L to S T, S K to S N, I L to H T, V T + T c to V T, and K c to N T,

are

are nearer to that of equality than by any given or aſſigned magnitudes of the ſame kind, or than any given or aſſigned ratios; and the ratios which V K, K L, V T + T L, S L, S K, I L, V T c, V T + T c, K c, have reſpectively to the given magnitude V G, are nearer to the ratios which V N, N T, V T, S T, S N, H T, V T, N T, have reſpectively to the ſame given magnitude V G, than by any given or aſſigned magnitudes of the ſame kind, or than any given or aſſigned ratios. Alſo T L, T c, and the chord T c, as well as their ratios, are nearer to equality than by any given or aſſigned magnitudes, or than any given or aſſigned ratios; and P L, P c, as well as their ratios, are nearer to equality than by any given or aſſigned magnitude, or than any given or aſſigned ratio.

Wherefore, as the direction of the curve and tangent at the point (T) is the ſame, the antecedentals of the abſciſs V N, the ſemi-ordinate N T, and the curve V T, are to each other as T P, P L, and T L, or as the ſub-tangent, the ſemi-ordinate, and tangent, reſpectively.

In like manner, if V K be to V N in a relation nearer to that of equality than by any given or aſſigned magnitude of the ſame kind, the areas V T c K V, G V T c I G are to the areas V T N V, G V T H G reſpectively, in relations nearer to that of equality than by any given or aſſigned magnitudes of the ſame kind, or than any given or aſſigned ratios; and the antecedentals of the areas V T N V, V N H G, are to each other as T N to N H.

In the ſame manner does it appear, that the antecedental of the ſolid ſegment, (of one-half of which V T N repreſents a ſection through the axis or abſciſs V N) is as the area of the ſection through T N, at right angles to V N, and the antecedental of its ſurface as the perimeter of this ſection, and the antecedental of the curve V T. Whence we get,

1ſt. The ratio of the antecedental of the abſciſs to that of the correſponding ſemi-ordinate, equal to the ratio of the ſub-tangent to the ſemi-ordinate.

2dly, The ratio of the antecedental of the abſciſs to that of the curve, equal to the ratio of the ſub-tangent to the tangent.

3dly, The ratio of the antecedental of the ſemi-ordinate to that of the curve, equal to the ratio of the ſemi-ordinate to the tangent.

4thly, The ratios of the antecedentals of the areas V T N, V H, to the ſquare on the given line V G; or B, compounded of the ratios of T N and N H to B, and of that of the antecedental of the abſciſs to B.

5thly, The ratio of the antecedental of the ſurface of the ſolid ſegment (of which V T N is the ſemi-ſection), to the ſquare on the given line B, compounded of the ratio of the perimeter of the ſection through T N at right angles to V N to B, and of that of the antecedental of the curve V T to B.

6thly, The ratio of the antecedental of the ſaid ſolid ſegment to the cube on the given line B, compounded of the ratio of the ſaid ſection through T N, to the ſquare on B, and of that of the antecedental of the abſciſs V N to B.

The foregoing method, in which every expreſſion is truly and ſtrictly geometrical, is founded on principles frequently made uſe of by the ancient Geometers, principles admitted into the very firſt Elements of Geometry, and repeatedly uſed by EUCLID himſelf. As it is a branch of general geometrical proportion, or univerſal compariſon, and is derived from an examination of the Antecedents of Ratios, having given conſequents and a given ſtandard of compariſon, in the various degrees of augmentation and diminution, they undergo by compoſition and decompoſition, according to the method I delivered geometrically in 1776 and 1777, and which, I am not ſenſible, that any perſon ever pointed out before, I have called it the Antecedental Calculus. It firſt occurred to me in 1774. As it is purely geometrical, and perfectly ſcientific, I have ſince that time always made uſe of it inſtead of the Fluxionary and Differential Calculi, which are merely arithmetical. Its principles are totally unconnected with the ideas of Motion and Time, which, ſtrictly ſpeaking, are foreign to pure Geometry and abſtract Science, though in mixed Mathematics and Natural Philoſophy they are equally applicable to every inveſtigation, involving the conſideration of either, with the two numerical methods juſt mentioned. And, as many ſuch inveſtigations require compoſitions and decompoſitions of ratios extending greatly beyond the triplicate and ſub-triplicate, this Calculus in all of them furniſhes every expreſſion in a ſtrictly geometrical form. The ſtandards of compariſon in it may be any magnitudes whatever, and are, of courſe, indefinite and innumerable; and the conſequents of the ratios, compounded or decompounded, may be either equal or unequal, homogeneous or heterogeneous. In the fluxionary and differential Methods, on the other hand, 1, or unit, is not only the invariable ſtandard of compariſon, but alſo the conſequent of every ratio compounded or decompounded. Thus, for inſtance, $nx^{n-1}\dot{x}$, when n is greater than 3, is not a geometrical expreſſion, but an arithmetical one, having to 1, or unit, the ratio, which ariſes by compounding the ratio of $\dot{x}$ to 1 with that of n to 1, and the ratio having to that of x to 1 the ratio of $n-1$ to 1. Alſo $nx^{n-1}y^{m}\dot{x}+my^{m-1}x^{n}\dot{y}$, when $n+m$ is greater than 3, cannot be a geometrical expreſſion, but is a number ariſing from the compoſition of the ratios $\dot{x}$ to 1, n to 1, the ratio having to that of x to 1 the ratio of $n-1$ to 1, and the ratio having to that of y to 1 the ratio of m to 1, together with the number ariſing from the compoſition of the ratios $\dot{y}$ to 1, m to 1, the ratio having to that of y to 1 the ratio of $m-1$ to 1, and the ratio having to that of x to 1 the ratio of n to 1. To multiply examples is needleſs. It appears from the writings of that truly great man, Sir ISAAC NEWTON, that he introduced into Geometry the idea of Velocity, chiefly with the view of avoiding the exceptionable doctrine of Indiviſibles, and conſidered lines, ſurfaces, and ſolids, as generated by the motions of points, lines, and ſurfaces, inſtead of being made up of them, or formed by the appoſition of infinite numbers of indiviſible parts. And in his Doctrine of prime and ultimate Ratios, he has recourſe to the idea of time, which, however, there was certainly no neceſſity for. And I am perfectly ſatiſfied, that had

had this great Man discovered the possibility of investigating a general Geometrical Method of reasoning, without introducing the ideas of Motion and Time applicable to every purpose, to which his Doctrines of Fluxions and prime and ultimate Ratios can be applied, he would have greatly preferred it, since Time and Motion have no natural or inseparable connection with pure Mathematics. The fluxionary and differential Calculi are only branches of general arithmetical proportion, and the expressions in them are numerical.

Thus have I delivered, with, I hope, sufficient conciseness, the general Geometrical Principles of what I call the Antecedental Calculus; and, I flatter myself, with perspicuity enough at the same time to render it plain and intelligible to readers of even ordinary capacities. This is all I intended. For, had I ever so much inclination, I have not leisure at present to communicate to the Public the fiftieth part of what I have by me in manuscript on the subject. I shall only observe, that the Theorems in the Universal Comparison furnish great variety of ways of expressing Antecedentals very different, abstracting even from their geometrical form, from any that have been hitherto used for Fluxions or Differentials, and of the greatest convenience in many difficult investigations. It may not perhaps be improper to add, that, if to the expressions delivered above for the excess of the magnitude, which has to B a ratio, having to the ratio of A + N to B, the ratio of R to Q, above the magnitude, which has to B a ratio, having to the ratio of A to B the same ratio of R to Q; and for the excess of the magnitude, which has to B a ratio, having to the ratio of A to B the ratio of R to Q above the magnitude, which has to B a ratio, having to the ratio of A — N to B the ratio of R to Q, be prefixed the magnitude, which has to B a ratio, having to the ratio of A to B the ratio of R to Q, we get a geometrical Binomial, of which, when it is supposed to become numerical, the famous Binomial Theorem of Sir ISAAC NEWTON is only a particular case. This derivation of it will, perhaps, be more agreeable and intelligible to some readers, than that given in the Universal Comparison itself.

Whoever can judiciously apply the Theorems in the Universal Comparison to the Summation of Series, will also find, that all that has been done in this branch of science, is but very limited in respect of the endless classifications, which they furnish, in forms too strictly geometrical. But the illustration of this extensive subject would carry me greatly beyond the limits I prescribed to myself in this concise publication.

OF INCREMENTS.

Page 9. Line 4. Universal Comparison.

Succeeding Values of the Antecedent.

$$A = A.$$

$$\underset{1}{A} \text{ or } A.\frac{A}{B} = A + A.\frac{A-B}{B} \text{ or } A + A.\frac{D}{B} \text{ (putting D for A−B)}$$

$$\underset{2}{A} \text{ or } A.\frac{A^2}{B^2} = A + 2A.\frac{D}{B} + A.\frac{D^2}{B^2}.$$

$$\underset{3}{A} \text{ or } A.\frac{A^3}{B^3} = A + 3A.\frac{D}{B} + 3A.\frac{D^2}{B^2} + A.\frac{D^3}{B^3}.$$

$$\underset{4}{A} \text{ or } A.\frac{A^4}{B^4} = A + 4A.\frac{D}{B} + 6A.\frac{D^2}{B^2} + 4A.\frac{D^3}{B^3} + A.\frac{D^4}{B^4}.$$

&c. &c.

1st Differences.	2d Differences.	3d Differences.
$A.\frac{D}{B}$.	$A.\frac{D^2}{B^2}$.	$A.\frac{D^3}{B^3}$.
$A.\frac{D}{B} + A.\frac{D^2}{B^2}$.	$A.\frac{D^2}{B^2} + A.\frac{D^3}{B^3}$.	$A.\frac{D^3}{B^3} + A.\frac{D^4}{B^4}$.
$A.\frac{D}{B} + 2A.\frac{D^2}{B^2} + A.\frac{D^3}{B^3}$.	$A.\frac{D^2}{B^2} + 2A.\frac{D^3}{B^3} + A.\frac{D^4}{B^4}$.	&c.
$A.\frac{D}{B} + 3A.\frac{D^2}{B^2} + 3A.\frac{D^3}{B^3} + A.\frac{D^4}{B^4}$.	&c.	
&c.		

Hence it is manifeſt, that the firſt, ſecond, third, &c. Increment of the Antecedent A, is the firſt term of the firſt, ſecond, third, &c. differences between the ſaid antecedent A and its ſucceeding values; a well-known and fundamental principle in the Doctrine of Increments. But here the expreſſions are all geometrical, and the Increments are magnitudes of the ſame kind with A, and with each other. When the ratios compounded or decompounded, are unequal Theorems 1, and 2, Univerſal Compariſon are applicable to this doctrine. Indeed, the whole geometrical Rationalia of the Method of Increments are ſo manifeſtly derivable from the Formulæ in that performance, that to dwell longer on the ſubject would be ſuperfluous. This method therefore, ſcientifically conſidered, is a branch of general proportion.

OF

OF THE MEASURES OF RATIOS.

It is evident that the ſecond term in Formula 5th, Theorem 3, Univerſal Compariſon is as the meaſure of (or to ſpeak more correctly as the quantity of, or degree of magnitude in,) the ratio, which the magnitude has to B, that has it to ~~A~~ a ratio, having to that of A to B the ratio of R to Q. For if Q = 1 and R = 1, 2, 3, 4, &c, reſpectively, $\frac{R}{Q}.\ \overline{A-B}$ is ſucceſſively expreſſed by $A-B$, $2.\ \overline{A-B}$, $3.\ \overline{A-B}$, $4.\ \overline{A-B}$, &c. and if R = 1, and Q = 1, 2, 3, 4, &c, by $A-B$, $\frac{A-B}{2}$, $\frac{A-B}{3}$, $\frac{A-B}{4}$, &c. Now, if with the ratios which theſe magnitudes reſpectively have to B, be compounded, the ratio of any given magnitude M to A, the antecedents to the conſequent B, will ſtill be as the quantities of the ratios, &c, expreſſed thus———

$$\left.\begin{array}{c} M - M.\frac{B}{A} \\ 2M - 2M.\frac{B}{A} \\ 3M - 3M.\frac{B}{A} \\ \text{\&c. to} \\ \frac{R}{Q}.M - \frac{R}{Q}.\frac{M.B}{A} \end{array}\right\} \begin{array}{l} \text{And if for } B-A \text{ there be} \\ \text{ſubſtituted } D, \text{ they become,} \\ \text{if } M = B, \end{array} \left\{\begin{array}{c} B.\frac{D}{A}. \\ 2B.\frac{D}{A}. \\ 3B.\frac{D}{A}. \\ \text{\&c. to} \\ \frac{R}{Q}.B.\frac{D}{A}. \end{array}\right.$$

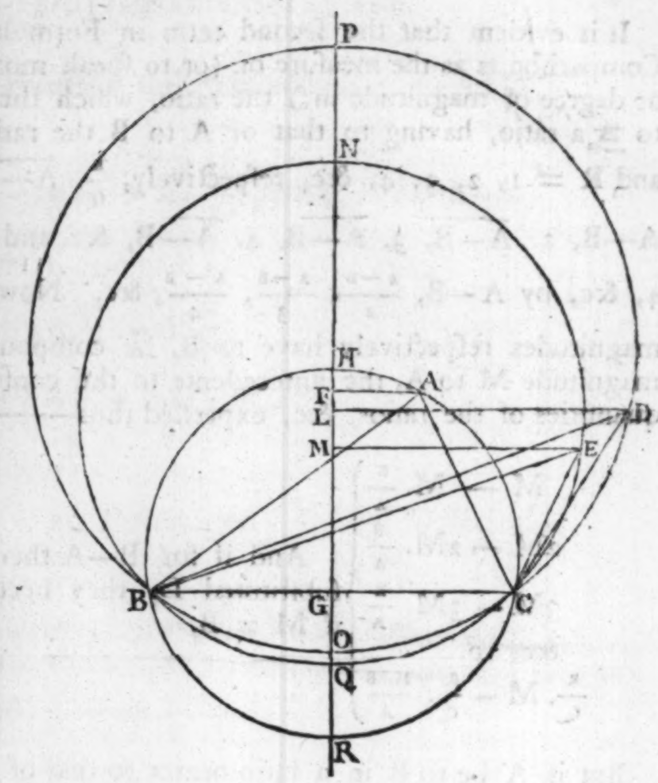
But if A be to B in a ratio nearer to that of equality than by any given or aſſigned magnitude of the ſame kind, or than any given or aſſigned ratio, the ratio of A to $A-B$, is greater than any given or aſſigned ratio; and the meaſure of the ratio of A to B to the modulus B, or ſtandard of meaſurement, is in this caſe expreſſed by $B.\frac{A-B}{A}$, or $B.\frac{\dot{A}}{A}$, and by $M.\frac{\dot{A}}{A}$ to the modulus or ſtandard M. Mr. Cotes, in his *Harmonia Menſurarum*, has conſidered this buſineſs in a merely arithmetical light, and with ſcarce any regard to the geometrical management of ratios. For his expreſſion $\frac{PQ}{AP}$ is not a geometrical one, and he uſes the very language of Indiviſibles.

I ſhall now give the Conſtructions of a few Problems in ſolid Geometry, by common or plane Geometry.

Problem 1. On any given right line as a baſe to conſtitute a triangle ſuch, that the cubes on the two other ſides ſhall together be equal to the cube on the ſaid given right line or baſe.

CON-

CONSTRUCTION, Fig. 1.



Let BC be the given right line. Let it be bisected in G, and through G let the indefinite right line PGR be drawn at right angles to it. Take GH equal to BC $\times \frac{3\sqrt{5}}{2\sqrt{31}}$ and about the triangle BHC describe the circle BHCR. Then take GF equal to BC $\times \frac{\sqrt{5 \cdot 31}}{24}$. Draw FA parallel to BC to meet the circle in A, and join BA, AC. Then BAC is the triangle required.

Problem 2. On any given right line as a base to constitute a triangle such, that the cubes on the other two sides shall together be equal to twice the cube on the given right line or base.

CONSTRUCTION, Fig. 1.

Let BC be the given right line as in Problem 1. Take GN equal to BC $\times \frac{\sqrt{15}}{\sqrt{11}}$, and about the triangle BNC describe the circle BNCQ. Then take GM equal to BC $\times \frac{4\sqrt{11}}{9\sqrt{15}}$, and draw ME parallel to BC to meet the circle in E. Join BE, EC, and BEC is the triangle required.

Problem 3. On any given right line as a base to constitute a triangle such, that the cubes on the other two sides shall together be equal to thrice the cube on the said given right line or base.

CON-

CONSTRUCTION, Fig. 1.

Take GP equal to $BC \times \frac{3}{2}$, and about the triangle BPC describe the circle BPCO. Then take GL equal to one-half of BC, and draw LD parallel to BC meeting the circle in the point D. Join BD, DC, and BDC is the triangle required.

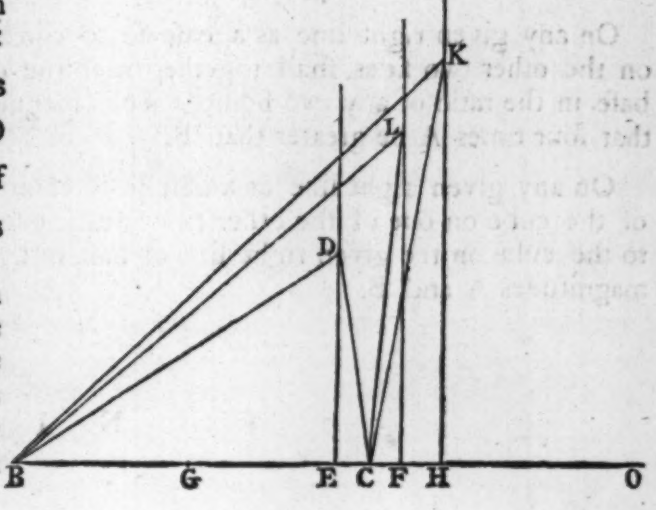
With equal facility could I proceed constructing Problems of this sort, *ad infinitum*, or indefinitely. Every such Problem is constructible by plane geometry a variety of ways, and between every two of them an indefinite number of others may be constructed. Thus, for instance, between Problems 1 and 2, there may be constructed on BC a triangle such, that the cubes on the other two sides shall together be to the cube on BC in the ratio of any two magnitudes A and B of the same kind, provided only that the ratio of A to B be between a ratio of equality and that of 2 to 1; and between Problems 2 and 3, there may be constructed on BC a triangle such, that the cubes on the other two sides shall together be to the cube on BC in the ratio of any two homogeneous magnitudes A and B, provided only that the ratio of A to B be between the ratios of 2 to 1, and 3 to 1, or be greater than the ratio of 2 to 1, and less than that of 3 to 1.

The taking of such expressions as $BC \times \frac{\sqrt{5 \cdot 31}}{24}$, is much facilitated by some Propositions in a Paper of mine, on the Division of Right Lines, Surfaces, and Solids, &c. published in the Philosophical Transactions, Vol. 66.

Problem 4. On any given right line as a base to constitute a triangle such, that the cube on one of the other two sides shall exceed the cube on the other, by the cube on the given right line or base.

CONSTRUCTION, Fig. 2.

Let BC be the given right line, and let it be bisected in G. Take GE equal to $BC \times \frac{\sqrt{31}}{8\sqrt{3}}$; draw ED at right angles to BC or BO, and take ED equal to $BC \times \frac{\sqrt{19}}{8}$. Then if BD, DC be joined, BDC is the triangle required.



Problem. 5. On any given right line as a base to constitute a triangle such, that the cube on one of the other two sides shall exceed the cube on the other, by twice the cube on the given right line or base.

CON-

CONSTRUCTION, Fig. 2.

Take GF to GE as $\sqrt{63}$ to $\sqrt{31}$, draw FL at right angles to BO, and take FL to ED as $\sqrt{51}$ to $\sqrt{19}$. Then BLC is the triangle required.

Problem 6. On any given right line as a base to constitute a triangle such, that the cube on one of the other two sides shall exceed the cube on the other, by thrice the cube on the given right line or base.

CONSTRUCTION, Fig. 2.

Take GH to GE as $\sqrt{95}$ to $\sqrt{31}$, draw HK at right angles to BO, and take HK to ED as $\sqrt{83}$ to $\sqrt{19}$. Then BK~~H~~ is the triangle required. BKC

In this manner, and with equal facility, could I proceed indefinitely constructing such Problems by plane Geometry, had I only leisure sufficient for this purpose. I have been for years in possession of geometrical principles, by which millions of them can be constructed. Between every two of these also, an indefinite number of others may also be constructed. Thus, for instance, between Problems 4 and 5, there may be constructed on BC a triangle such, that the excess of the cube on one of the other two sides above the cube on the other, shall be to the cube on BC in the ratio of any two homogeneous magnitudes A and B, provided only, that the ratio of A to B be between a ratio of equality and that of 2 to 1; and so on without end.

This is a geometrical field, which neither the Ancients nor Moderns seem so much as even to have looked into, unlimited both as to extent and variety; and it furnishes the means of enriching pure Geometry infinitely more than all that has been written by Sir Isaac Newton and other ingenious men on curves and lines of different orders. To give the reader some idea of its extent, I will conclude this short performance with the two following Problems:

On any given right line as a base so to constitute a triangle, that the cubes on the other two sides shall together be to the cube on the given right line or base in the ratio of any two homogeneous magnitudes A and B, provided only, that four times A be greater than B.

On any given right line as a base so to constitute a triangle, that the excess of the cube on one of the other two sides above the cube on the other, shall be to the cube on the given right line or base in the ratio of any two homogeneous magnitudes A and B.

F I N I S.